Lecture 23

In this lecture we continue our study of predicate formulae involving two quantifiers.

Suppose $P(x, y)$ means $x \geqslant y$.

Let the domain of interpretation D be the set $\mathbb{N} \setminus \{0\}$ of all positive integers.

$\forall x \, \forall y \; P(x, y)$

$\forall x \, \exists y$ "

$\exists x \, \forall y$ "

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$\exists y \, \forall x$ "

$$\boxed{\begin{array}{l} \mathbb{N} \setminus \{0\} = \\ \{1, 2, 3, \ldots\} \end{array}}$$

1 $\forall x \forall y\, P(x,y)$

For all x and (for all) y, $\underbrace{P(x,y)}_{x \geq y}$ is true

This says that no matter which numbers x and y we choose from $\mathbb{N} \setminus \{0\}$, it will always happen that $x \geq y$.

Is this true?

No. For example, $x = 1$ and $y = 2$.

2 $\forall x \exists y\, P(x,y)$

For all x there exists y such that $x \geq y$.

Here, y can depend on x. A different choice of x may lead to a different value of y.

Is this true?

$\boxed{\{1, 2, 3, \ldots\}}$

Yes. For example, given x we can take $y = x$. Then $x \geq y$.

Or, we could take $y = 1$ when $x = 1$, and take $y = x - 1$ for all other values of x.

Or we could take $y = 1$ always.

(3) <u>$\exists x \forall y \; P(x, y)$</u>

There exists x <u>such that</u> for all y, $x \geq y$.

This says that x is a constant, and every choice of y makes $x \geq y$.

Is this true?

No. There is no such constant. (It would have to be the biggest integer — the largest element of $\mathbb{N} \setminus \{0\}$. But this set has no largest element.)

(4) $\underline{\exists x \, \exists y \; P(x,y)}$

There exists x and there exists y such that $x \geq y$.

Is this true?

Yes. E.g., take $x = 2$ and $y = 1$.

(5) $\underline{\forall y \, \exists x \, P(x,y)}$

For all y there exists x <u>such that</u> $x \geq y$.

This says that for every choice of y it's possible to find an x which is $\geq y$.

Is this true?

Yes. E.g., put $x = y+1$. (Or take $x=y$.)

(6) $\underline{\exists y \, \forall x \, P(x,y)}$

There exists y <u>such that</u> for all x, $x \geq y$.

This says that there is a constant y which is less than or equal to all values of x.

Is this true?

Yes — $y = 1$ has this property. It's the smallest element of the set.

Exercises

(a) How do the results change if D changes to a finite set of numbers?

(b) How do the results change if D changes from $\mathbb{N} \setminus \{0\}$ to $\mathbb{Z}$?

(c) How do the results change if $P(x, y)$ changes to "$x > y$"?